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Green's Function for a Spherical Cell*

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INTRODUCTION

Electrical potential problems occurring in the electrophysiology of nerve and muscle differ from those usually considered in electrical theory for two reasons: first, the relation between current flow and polarization across a membrane of tissues is nonlinear, and second, because the interior of the tissue is not strictly equipotential.

Weinberg [1] formulates these problems and defines a Green's function which is suitable for their solution. He supposes u_e and u_i are the steady state potentials outside and inside a tissue covered by a closed polarized membrane, S. Then u_e and u_i are both harmonic functions which satisfy the non-linear boundary condition at S: $k_e \partial u_e/\partial n_e = -k_i\, \partial u_i/\partial n_i = F(u_e - u_i)$, where k_e and k_i are the conductivities of the external and internal media, $F(u_e - u_i)$ is some function of the membrane potential difference which determines the current flow across the membrane, and n_e and n_i are the outwardly and inwardly drawn normals on S. In finding u_e and u_i by using Green's Theorem Weinberg introduces two harmonic functions, H_e and H_i, the first defined in the exterior and the second in the interior. These harmonic functions have the property:

$$\partial G_e^e/\partial n_e = -k_i/k_e\, \partial G_i^e/\partial n_i = G_e^e - G_i^e \text{ on } S, \tag{1}$$

where

$$G_e^e(x, y, z; \xi, \eta, \zeta) = 1/\rho_e + H_e(x, y, z; \xi, \eta, \zeta); \quad \begin{matrix}(x, y, z)\\ (\xi, \eta, \zeta)\end{matrix} \text{ in exterior,} \tag{2}$$

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and

$$G_i^e(x, y, z; \xi, \eta, \zeta) = 1/\rho_e + H_i(x, y, z; \xi, \eta, \zeta); \quad (x, y, z) \text{ in exterior}, \quad (\xi, \eta, \zeta) \text{ in interior}. \tag{3}$$

The distance ρ_e from an exterior point $P(x, y, z)$ to a variable point $P'(\xi, \eta, \zeta)$ is $\rho_e^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2$. The reciprocal distance $1/\rho_e$ has a pole in the exterior at the point P, but is a continuous function everywhere in the interior.

Case of a Spherical Cell

We would like to find the Green's functions, G_e^e and G_i^e, for a spherical cell of radius R with a polarized membrane subject to the boundary conditions (1). Since we are dealing with a spherical cell we will change to spherical coordinates in our problem.

We look for a function $G(r', \theta', \phi'; r, \theta, \phi)$ such that

$$G = \begin{cases} G_e^e(r', \theta', \phi'; r, \theta, \phi) = 1/\rho_e + H_e(r', \theta', \phi'; r, \theta, \phi) & r' > R, r > R \\ G_i^e(r', \theta', \phi'; r, \theta, \phi) = 1/\rho_e + H_i(r', \theta', \phi'; r, \theta, \phi) & r' < R, r > R \end{cases}$$

with H_e and H_i harmonic in r', θ', ϕ' variables and with the boundary condition:

$$\begin{aligned} \partial G_e^e/\partial r'|_{r'=R} &= k\partial G_i^e/\partial r'|_{r'=R} \\ &= G_e^e(R, \theta', \phi'; r, \theta, \phi) - G_i^e(R, \theta', \phi'; r, \theta, \phi), \end{aligned} \tag{4}$$

where $k = k_i/k_e$. We have

$$\frac{\partial}{\partial r'} \frac{1}{(r^2 + r'^2 - 2rr' \cos \gamma)^{1/2}} \bigg|_{r'=R} = -(R - r \cos \gamma)/(R^2 + r^2 - 2rR \cos \gamma)^{3/2},$$

where $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi')$. The boundary condition then becomes, in terms of the H's:

$$\begin{aligned} &H_e(R, \theta', \phi'; r, \theta, \phi) - H_i(R, \theta', \phi'; r, \theta, \phi) \\ &\quad = -(R - r \cos \gamma)/(R^2 + r^2 - 2rR \cos \gamma)^{3/2} + \partial H_e/\partial r'|_{r'=R} \end{aligned} \tag{5}$$

$$\begin{aligned} &H_e(R, \theta', \phi'; r, \theta, \phi) - H_i(R, \theta', \phi'; r, \theta, \phi) \\ &\quad = -k(R - r \cos \gamma)/(R^2 + r^2 - 2rR \cos \gamma)^{3/2} + k\partial H_i/\partial r'|_{r'=R}. \end{aligned} \tag{6}$$

Now H_e is to be harmonic outside the sphere and finite at $r' = \infty$, and H_i is to be harmonic inside the sphere and finite at $r' = 0$. By separation of variables we find the following representations [2] for these:

$$H_e(r', \theta', \phi'; r, \theta, \phi) = \sum_{m,n} [A^{(e)}_{mn} Y^e_{mn}(\theta', \phi') + B^{(e)}_{mn} Y^0_{mn}(\theta', \phi')](R/r')^{n+1} \tag{7}$$

$$H_i(r', \theta', \phi'; r, \theta, \phi) = \sum_{m,n} [A^{(i)}_{mn} Y^e_{mn}(\theta', \phi') + B^{(i)}_{mn} Y^0_{mn}(\theta', \phi')](r'/R)^n, \tag{8}$$

where $Y^e_{mn} = \cos(m\phi)P_n{}^m(\cos\theta)$ and $Y^0_{mn} = \sin(m\phi)P_n{}^m(\cos\theta)$. The A's and B's are functions of r, θ, and ϕ only. We substitute (7) and (8) into (5) and (6) to obtain:

$$\begin{aligned}\sum_{m,n} \{[A^{(e)}_{mn} &- A^{(i)}_{mn} + (n+1)R^{-1}A^{(e)}_{mn}]Y^e_{mn}(\theta', \phi') \\ &+ [B^{(e)}_{mn} - B^{(i)}_{mn} + (n+1)R^{-1}B^{(e)}_{mn}]Y^0_{mn}(\theta', \phi')\} \\ &= -(R - r\cos\gamma)/(R^2 + r^2 - 2rR\cos\gamma)^{3/2}\end{aligned} \tag{9}$$

and

$$\begin{aligned}\sum_{m,n} \{[A^{(e)}_{mn} &- A^{(i)}_{mn} - knR^{-1}A^{(i)}_{mn}]Y^e_{mn}(\theta', \phi') \\ &+ [B^{(e)}_{mn} - B^{(i)}_{mn} - knR^{-1}B^{(i)}_{mn}]Y^0_{mn}(\theta', \phi')\} \\ &= -k(R - r\cos\gamma)/(R^2 + r^2 - 2rR\cos\gamma)^{3/2}.\end{aligned} \tag{10}$$

We now introduce the expansion of the right-hand side in spherical harmonics:

$$\begin{aligned}-(R &- r\cos\gamma)/(R^2 + r^2 - 2rR\cos\gamma)^{3/2} \\ &= \sum_{m,n} [\alpha_{mn} Y^e_{mn}(\theta', \phi') + \beta_{mn} Y^0_{mn}(\theta', \phi')] \\ &= \sum_{m,n} [\alpha_{mn}\cos(m\phi')P_n{}^m(\cos\theta') + \beta_{mn}\sin(m\phi')P_n{}^m(\cos\theta')].\end{aligned} \tag{11}$$

We would like to find the coefficients α_{mn} and β_{mn}. The series expansion of $1/\rho_e$ in spherical harmonics is known [2, p. 69] so we will differentiate this expansion with respect to r' and evaluate it at $r' = R$ to obtain:

$$\left.\frac{\partial}{\partial r'}\frac{1}{\rho_e}\right|_{r'=R} = \sum_{m,n} \epsilon_m (n-m)!/(n+m)!\, P_n{}^m(\cos\theta')P_n{}^m(\cos\theta)\cos[m(\phi - \phi')] \times \begin{Bmatrix} -(n+1)r^n R^{-(n+2)}, & R > r \\ nR^{n-1}r^{-(n+1)}, & r > R \end{Bmatrix} \tag{12}$$

where $\epsilon_m = 1$ when $m = 0$, $= 2$ when $m > 0$ (Neumann factor). We expand the factor $\cos [m(\phi - \phi')]$ in the usual way and when we equate series (11) with series (12) we see that

$$\alpha_{mn} = \epsilon_m (n-m)!/(n+m)!\, P_n{}^m(\cos\theta)\cos(m\phi) \times \begin{cases} -(n+1) r^n R^{-(n+2)}, & R > r \\ n R^{n-1} r^{-(n+1)}, & r > R \end{cases} \tag{13}$$

and β_{mn} is similar to α_{mn} except that $\sin(m\phi)$ is substituted for $\cos(m\phi)$.

Upon comparing coefficients of $\cos(m\phi')P_n{}^m(\cos\theta')$ and $\sin(m\phi')P_n{}^m$ $(\cos\phi')$ in (9) and (10) we find

$$\begin{aligned} [1+(n+1)R^{-1}]A^{(e)}_{mn} - A^{(i)}_{mn} &= \alpha_{mn} \\ [1+(n+1)R^{-1}]B^{(e)}_{mn} - B^{(i)}_{mn} &= \beta_{mn} \\ A^{(e)}_{mn} - (1+knR^{-1})A^{(i)}_{mn} &= k\alpha_{mn} \\ B^{(e)}_{mn} - (1+knR^{-1})B^{(i)}_{mn} &= k\beta_{mn}\,. \end{aligned} \tag{14}$$

We solve these equations for the A's and B's and substitute into (7) and (8), and write

$$\begin{aligned} H_e = \sum_{m,n} \{\{k + [1+knR^{-1}][k-1+k(n+1)R^{-1}]/1 \\ - [1+(n+1)R^{-1}][1+knR^{-1}]\} \\ \times \{\alpha_{mn} Y^e_{mn}(\theta',\phi') + \beta_{mn} Y^0_{mn}(\theta',\phi')\}\}(R/r')^{n+1}, \end{aligned} \tag{15}$$

$$\begin{aligned} H_i = \sum_{m,n} \{\{[k-1+k(n+1)R^{-1}]/1 - [1+(n+1)R^{-1}][1+knR^{-1}]\} \\ \times \{\alpha_{mn} Y^e_{mn}(\theta',\phi') + \beta_{mn} Y^0_{mn}(\theta',\phi')\}\}(r'/R)^n. \end{aligned} \tag{16}$$

Thus the Green's functions G for a spherical cell subject to the boundary conditions (1) are:

$$\begin{aligned} G_e{}^e = \sum_{m,n} \epsilon_m (n-m)!/(n+m)!\, P_n{}^m(\cos\theta')P_n{}^m(\cos\theta)\cos[m(\phi-\phi')] \\ \times \begin{cases} r^n/R^{n+1}, & R > r \\ R^n/r^{n+1}, & r > R \end{cases} \\ + \sum_{m,n} \{\{k+[1+knR^{-1}][k-1+k(n+1)R^{-1}]/1 - [1+(n+1)R^{-1}][1+knR^{-1}]\} \\ \times \{\alpha_{mn} Y^e_{mn}(\theta',\phi') + \beta_{mn} Y^0_{mn}(\theta',\phi')\}\}(R/r')^{n+1} \end{aligned} \tag{17}$$

and

$$G_i^e = \sum_{m,n} \epsilon_m (n-m)!/(n+m)!\, P_n{}^m(\cos\theta') P_n{}^m(\cos\theta) \cos[m(\phi-\phi')] \times \begin{cases} r^n/R^{n+1}, & R > r \\ R^n/r^{n+1}, & r > R \end{cases}$$
$$+ \sum_{m,n} \{\{[k-1+k(n+1)R^{-1}]/1 - [1+(n+1)R^{-1}][1+knR^{-1}]\} \times \{\alpha_{mn} Y^e_{mn}(\theta', \phi') + \beta_{mn} Y^0_{mn}(\theta', \phi')\}\}(r'/R)^n. \tag{18}$$

A particular application of this analysis must wait until further knowledge is available concerning the exact form of the boundary function F.

References

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